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The measurement of income mobility: A partial ordering approach★

Tapan Mitra[1] **and Efe A. Ok**[2]

[1] Department of Economics, Cornell University, Ithaca, NY 14853, USA

[2] Department of Economics, New York University, 269 Mercer St., New York, NY 10003, USA

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Summary. Given a set of longitudinal data pertaining to two populations, a question of interest is the following: Which population has experienced a greater extent of *income mobility*? The aim of the present paper is to develop a systematic way of answering this question. We first put forth four axioms for income movement-mobility indices, and show that a familiar class of measures is characterized by these axioms. An unambiguous (partial) ordering is then defined as the intersection of the (complete) orderings induced by the mobility measures which belong to the characterized class; a transformation of income distributions is "*more mobile*" than another if, and only if, the former is ranked higher than the latter for *all* mobility measures which satisfy our axioms. Unfortunately, our mobility ordering depends on a parameter, and therefore, it is not readily apparent how one can apply it to panel data directly. In the second part of the paper, therefore, we derive several sets of parameter-free necessary and sufficient conditions which allow one to use the proposed mobility ordering in making unambiguous income mobility comparisons in practice.

JEL Classification Numbers: D31, D63.

1 Introduction

Suppose we have observed the evolution of the income distributions of two different populations through time. Let us also assume that we have panel data at hand so that the individual income changes in both populations are known. One of the interesting questions that can be asked with such givens is

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Correspondence to: T. Mitra

the following: Which one of these populations has experienced a greater degree of *income mobility*?

This question has attracted numerous economists, and a number of methods to study the basic measurement problem have appeared in the literature. (See Fields and Ok, 1996a, for a recent survey.) Unfortunately, it seems fair to say that the related literature falls short of providing a unified way of measuring income mobility. This is, of course, in sharp contrast with the structurally similar problem of the measurement of income inequality where, in the light of several studies that followed the seminal contributions of Kolm (1969) and Atkinson (1970), the implementation of the (relative) Lorenz ordering emerged as a unifying theme.

The usual practice of income mobility measurement is by way of employing certain (descriptive) mobility indicators (like rank correlation, immobility ratio, average jump in rank, Hart's index, Maasoumi-Zandvakili index and Shorrocks' index).[1] However, more often than not, the used measures are not axiomatically examined; the generic approach is indeed remarked as being rather ad hoc (Cowell, 1985). Moreover, there does not exist a (descriptive) partial ordering (reminiscent of the Lorenz ordering) which lets us unambiguously rank transformations of income distributions on the basis of their mobility content. In this paper, therefore, we aim to supplement the existing theory of income mobility measurement both by axiomatically characterizing a class of (absolute) income mobility indices and by using this class to propose a partial mobility ordering which would allow us to make unambiguous income mobility comparisons.

Let us first clarify what we mean by "income mobility" in this paper. There are (at least) two distinct interpretations of the notion of income mobility (Bartholomew, 1982, pp. 24–30). The first is based on the notion of *temporal independence* as a proxy for the "equality of opportunity" concept (i.e., the extent to which personal characteristics rather than parental background determine monetary payoffs). By its very nature, however, such an interpretation of mobility requires an *inter*generational setting. In an *intra*generational framework, on the other hand, the second interpretation of income mobility, namely, the *aggregate income movements* (or the notion of *distributional change*) becomes more relevant.[2] In this paper, we shall focus on this latter interpretation which is clearly linked to the important welfare criterion of "lifetime income equality". By *income mobility*, therefore, we mean here the amount of *movement* involved in a given evolution of a particular income distribution. [Consequently, while our study parallels King (1983) and Cowell (1985) it is conceptually distinct from the mobility analyses of Shorrocks (1978) and Dardanoni (1993).] Having this interpretation in mind, we imagine a situation where an income distribution transforms to

[1] See Schiller (1977), Lillard and Willis (1978), Maasoumi and Zandvakili (1986) and Shorrocks (1978).

[2] See Fields and Ok (1996a) for a detailed discussion and comparison of these two aspects of mobility.

another and where we can identify the individual income changes. In such a context, by a *measure of income mobility*, we simply mean a method of aggregating the observed personal income differentials.

The first part of our analysis proceeds by postulating four axioms which appear quite reasonable to posit on an (absolute) income mobility measure. It is shown that these axioms characterize a rather familiar class of mobility (distributional change) indices. A generic member of this characterized class, D_n, is necessarily of the form

$$D_n(x,y) = \gamma\left(\sum_{k=1}^{n}|x_k - y_k|^{\alpha}\right)^{1/\alpha} \quad \text{for some } \gamma > 0 \text{ and } \alpha \geq 1 \ .$$

$D_n(x,y)$ is thought of as the total amount of absolute mobility observed in the process of "going" from the income distribution $x \in \mathbb{R}^n_+$ to the income distribution $y \in \mathbb{R}^n_+$.[3] As long as one finds our axioms appealing, therefore, (s)he would conclude that the process where an n-tuple x becomes y (denoted as $x \to y$) exhibits "*more income mobility*" than the process where an n-tuple z becomes w ($z \to w$) whenever

$$\sum_{k=1}^{n}|x_k - y_k|^{\alpha} \geq \sum_{k=1}^{n}|z_k - w_k|^{\alpha} \quad \text{for a certain choice of } \alpha \geq 1 \ .$$

Although this may be thought of as an interesting observation on its own right, we must note that the problem of ordinally comparing the levels of the absolute income mobility involved in $x \to y$ and $z \to w$ is not yet resolved, for the question remains: Which α value should one use?

There is, of course, a trivial way of overcoming the ambiguity surrounding the choice of the parameter α, namely, to demand the support of *all* $\alpha \geq 1$ values. This amounts to defining a partial ordering which lets us conclude unambiguously (with respect to the choice of α) that $x \to y$ involves more absolute income mobility than $z \to w$ if, and only if,

$$\sum_{k=1}^{n}|x_k - y_k|^{\alpha} \geq \sum_{k=1}^{n}|z_k - w_k|^{\alpha} \quad \text{for all } \alpha \geq 1 \ .$$ [4]

Given our axiomatic characterization, this partial ordering emerges as a useful (absolute) mobility ordering allowing us to rank transformations of

[3] When we say $x = (x_1, \ldots, x_n) \in \mathbb{R}^n_+$ goes to (or becomes) $y = (y_1, \ldots, y_n) \in \mathbb{R}^n_+$, we mean that the kth person's income has changed from x_k to y_k, $k = 1, \ldots, n$, in the time period considered.

[4] This approach is quite similar to that of the theory of income inequality measurement where S-concave inequality indices are obtained as the class of all "reasonable" inequality indices, and the problem of which S-concave index to use is partially resolved by demanding the agreement of *all* S-concave inequality measures. (Of course, this, in turn, lead us to the celebrated Lorenz ordering.) An analogous approach is also followed in Dardanoni (1993) where an interesting partial ordering of transition matrices in a Markovian model of social mobility is derived.

income distributions (according to their (descriptive) *movement* content) in a convincing way.

The problem is, of course, that such a partial ordering gives a continuous reference to the parameter α, and this makes it practically impossible to rank transformations $x \to y$ and $z \to w$ (except in trivial cases like $z = w$). The second part of our analysis is, therefore, devoted to determining parameter-free sets of (distinct) necessary and sufficient conditions for this ordering to be applicable. Although a parameter-free characterization of our mobility ordering does not seem to be within reach at present, we find that it is possible to obtain a number of interesting super- and subrelations of it (which are defined in a parameter-free manner). To illustrate, consider the hypothetical 3-person transformations reported in the following table:

Process		Personal income changes
I.	$(100, 40, 80) \to (100, 591, 80)$	(0, 551, 0)
II.	$(40, 400, 650) \to (590, 300, 750)$	(550, 100, 100)
III.	$(100, 20, 500) \to (615, 93, 573)$	(515, 73, 73)
IV.	$(440, 440, 30) \to (360, 950, 100)$	(80, 510, 70)
V.	$(670, 70, 100) \to (170, 170, 80)$	(500, 100, 20)
VI.	$(80, 600, 175) \to (160, 180, 115)$	(80, 420, 60)

By virtue of our necessity results (which provide us with certain superrelations of our mobility ordering), we are able to conclude that process I cannot be compared with any other process depicted above on the basis of an axiomatic approach. On the other hand, our sufficiency results enable us to unambiguously order the rest of the transformations reported above: Process II is "*more mobile*" than process III, process III is "*more mobile*" than process IV and so on. This example demonstrates that although it does not solve the problem at hand completely, the present development may still be useful in making unambiguous mobility comparisons between transformations of income distributions in some situations that may well arise in practice.

The paper is organized as follows. In Section 2, we introduce four properties which seem quite reasonable for an (absolute) total income mobility measure to satisfy. In this section, we also provide a characterization of the class of mobility measures that satisfy these four axioms. Section 3 defines our absolute income mobility ordering: an income distribution transformation is "*more mobile*" than another whenever *all* mobility measures that satisfy our axioms rank the former transformation higher than the latter. In Section 4 we derive a number of simple but useful necessary conditions which allow us to detect if our partial ordering fails to rank two given transformations. Section 5 deals with the converse question by obtaining several sufficiency conditions. These conditions are easy to check, and thus, they may turn out useful in empirical applications. Section 6 presents our concluding comments.

2 A class of income mobility measures

We consider $\mathbb{R}^n_+$ as the space of all income distributions with population $n \geq 1$. Thus, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n_+$ represents an income distribution where x_k is the level of income of the kth individual at a given point in time. Suppose that the kth agent's income has changed to y_k, $k \in \{1, \ldots, n\}$, or equivalently, that x evolves to $y \in \mathbb{R}^n_+$ in a given amount of time. We shall denote this transformation by $x \to y$. As noted in Fields and Ok (1996), asking how much mobility has taken place in this process might be rephrased as how much "apart" x and y have become for an appropriate distance function $D_n(.,.)$ on $\mathbb{R}^n_+$.[5] With this interpretation in mind, we view $D_n(x,y)$ as the (cardinal) level of *total absolute income mobility* that is observed in $x \to y$. The question is the following: What sort of distance functions D_n on $\mathbb{R}^n_+$ are appealing as a measure of total absolute income mobility? In what follows, we shall attempt to answer this question by using the axiomatic method.

Before proceeding to introduce our axioms, we emphasize that D_n is here interpreted as a measure of *total* income mobility in a population of n individuals. In other words, we wish D_n to never record a decrease in mobility if we include an additional person into the population who has experienced a positive income change in the time period under consideration. But this interpretation entails that the said measure cannot be considered as suitable in comparing the income mobilities of two populations of different sizes. This is, however, not a serious problem, for once one is convinced that D_n is a proper measure of total mobility for populations of size n, all we need to do is to use the *per capita* version of D_n which is naturally defined as

$$M_n(x,y) := \frac{D_n(x,y)}{n} \quad \text{for all } x,y \in \mathbf{R}^n_{++}, \;\; n \geq 1 \;\; .$$

When the sizes of the groups being analyzed vary, therefore, using M_n (as opposed to D_n) to make mobility comparisons is in nature of things. (The analogy with the familiar notions of total GNP and per capita GNP should be clear.) The task before us is thus discovering the acceptable form of D_n as a *total* measure of income mobility; this will readily provide us with a per capita measure.

Let $\mathscr{D}^n$ denote the class of all distance functions on $\mathbb{R}^n_+$, $n \geq 1$. Our first axiom reads as

Axiom LH: (**Linear Homogeneity**) Let $D_n \in \mathscr{D}^n$, $n \geq 1$. For all $x, y \in \mathbb{R}^n_+$ and $\lambda > 0$,

$$D_n(\lambda x, \lambda y) = \lambda D_n(x,y) \;\; .$$

In words, Axiom LH states that an equiproportional change in all income levels (both in the initial and final distributions) results in exactly the same

[5] See Dagum (1980), Ebert (1984) and Chakravarty and Dutta (1987) for a similar approach in the context of income inequality measurement. As noted earlier, the framework of Cowell (1985) where the measurement of distributional change is axiomatically studied is certainly very close to that of ours.

percentage change in the mobility measure, or put succinctly, D_n is *scale dependent*.

It must be clear that if $D_n \in \mathscr{D}^n$ satisfies Axiom LH, then it can only qualify for an *absolute* mobility measure as opposed to a *relative* mobility measure which must, by definition, be *scale invariant*. A relativist would therefore immediately object to Axiom LH. There are, however, at least two reasons why a researcher who is interested in relative mobility can still benefit from an absolute measure of mobility (which satisfies Axiom LH). First, comparing the absolute mobility content of a transformation along with its relative mobility can simply be revealing more information about the mobility of the process. Consider the processes $(1,2) \rightarrow (2,4)$ and $(10,20) \rightarrow (20,40)$, for instance. While a linearly homogeneous measure of mobility would indicate that the second transformation exhibits a higher *level* of (per capita and/or total) income growth (and this conclusion is hardly disputable), a relative measure would rightly indicate that these two processes are identical with respect to *percentage* income growth. We thus maintain that absolute and relative measures may be beneficially used to complement each other. (See Fields and Ok, 1996a, for more on this.) Second, a measure of (absolute) income mobility which satisfies Axiom LH can itself be used to determine the level of relative mobility in a given process. For instance, the mobility measure

$$P_n(x,y) := \frac{D_n(x,y)}{\sum_{k=1}^{n} x_k} \quad \text{for all } x,y \in \mathbf{R}^n_{++}, \ n \geq 1$$

would be scale invariant as long as D_n satisfies Axiom LH. (P_n can be thought of as a measure of *percentage* income mobility.) Consequently, we believe that studying total absolute income mobility measures could also prove useful in estimating the relative mobility content of distributional transformations.

Our next axiom is

Axiom TI: (**Translation Invariance**) Let $D_n \in \mathscr{D}^n$ and $\mathbf{1}_n := (1,\ldots,1) \in \mathbb{R}^n$, $n \geq 1$. For all $x,y \in \mathbb{R}^n_+$ and $\theta \in \mathbb{R}$ such that $x + \theta\mathbf{1}_n, y + \theta\mathbf{1}_n \in \mathbb{R}^n_+$,

$$D_n(x + \theta\mathbf{1}_n, y + \theta\mathbf{1}_n) = D_n(x,y) \ .$$

Axiom TI indicates that, given the amount of mobility found in going from one distribution to another, if the same amount is added to everybody's income in both the original and the final distributions, the new situation has the same mobility as the original one. This axiom guarantees formally that D_n is an *absolute* measure of mobility, and is thus related to Kolm's well-known *leftist* inequality criterion (cf. Kolm, 1976).

Of course, one may again object to Axiom TI from a relativist angle. Indeed, while an absolute mobility measure would see equal amount of mobility in the transformations $(2,2) \rightarrow (4,4)$ and $(100,100) \rightarrow (102,102)$, for instance, the latter process exhibits far less percentage movement than the former one. Our defense of Axiom TI is very similar to that of Axiom LH. Absolute mobility is something altogether different than relative mobility, a measure of it simply provides one with further information about the processes under study. In the case of the preceding example, for instance, we simply say that while there is the same level of absolute mobility in both transformations, there is more relative mobility in the former one. Moreover, as noted above, one may use a relative index induced by an absolute mobility measure to estimate relative mobility. For example, if D_2 is translation invariant, we have $P_2((2,2),(4,4)) > P_2((100,100),(102,102))$ as desired. We conclude then that there is reason to explore the implications of Axioms LH and TI for income mobility measures.

In passing, we stress that Axioms LH and TI are widely used in the literature on the theory of economic distances (see, e.g., Ebert, 1984, and Chakravarty and Dutta, 1987) and on the theory of aggregative compromise inequality measures (see, e.g., Blackorby and Donaldson, 1978, 1980, Eichhorn and Gehrig, 1982, and Ebert, 1988). The following is thus well-known.

Lemma 1: *$D_1 \in \mathscr{D}^1$ satisfies Axioms LH and TI if, and only if, for some $\gamma > 0$, $D_1(x,y) = \gamma|x-y|$ for all $x,y \geq 0$.*

Proof: If $D_1 \in \mathscr{D}^1$ satisfies Axioms LH and TI, then

$$D_1(x,y) = \begin{cases} D_1(x-y,0), & \text{if } x \geq y \\ D_1(y-x,0), & \text{if } x < y \end{cases} = D_1(|x-y|,0) = D_1(1,0)|x-y| \ ,$$

for any $x,y \geq 0$. The lemma readily follows from this observation. □

Our next axiom is fundamental to our present development.

Axiom D: (**Decomposability**) Let $D_n \in \mathscr{D}^n$, $n \geq 2$. For all $x,y \in \mathbb{R}^n_+$,

$$D_n(x,y) = G_n(D_1(x_1,y_1),\ldots,D_1(x_n,y_n))$$

for some symmetric, strictly increasing and continuous $G_n : \mathbb{R}^n_+ \to \mathbb{R}_+$.

Axiom D posits that the level of aggregate income mobility is a strictly monotonic function of the observed changes in the income levels of all agents (cf. Cowell, 1985). This function is further assumed to be symmetric to warrant the impartial treatment of the constituent individuals. Continuity of it is postulated as a weak regularity condition.

It is important to note that Axiom D forces one to view $D_n(.,.)$ as an *aggregation* of the distribution of individual income changes, and hence, it highlights the fact that our focus is not on the changes in the relative *ranks* of the agents.[6] It is in this sense our work is conceptually different than those of Plotnick (1982), King (1983) and Chakravarty (1984).

The following observation is straightforward.

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[6] For example, let $x = (1,2,5)$, $y = (1,4,5)$ and $z = (3,2,5)$. It is easy to see that $D_3(x,y) = D_3(x,z)$ for all $D_3 \in \mathscr{D}^3$ satisfying Axioms TI and D. However, $x \to y$ and $x \to z$ depict quite different situations with regard to changing ranks of the individuals. See Fields and Ok (1996) for a further discussion of this point.

Lemma 2: *Let $D_n \in \mathscr{D}^n$, $n \geq 2$, satisfy Axioms LH, TI and D. Then, for all $x, y \in \mathbb{R}^n_+$,*

$$D_n(x,y) = \gamma G_n(|x_1 - y_1|, \ldots, |x_n - y_n|)$$

for some symmetric, continuous and linearly homogeneous $G_n : \mathbb{R}^n_+ \to \mathbb{R}_+$ and for some $\gamma > 0$.

Proof: In view of Lemma 1 and Axiom D, we only need to demonstrate the linear homogeneity of G_n. Fix $n \geq 2$ and let $(a_1, \ldots, a_n) \in \mathbb{R}^n_+$ and $\lambda > 0$ be arbitrary. Choose any $x, y \in \mathbb{R}^n_+$ such that $|x_k - y_k| = a_k/\gamma$, $k = 1, \ldots, n$. By Axioms LH and D,

$$\begin{aligned} G_n(\lambda a_1, \ldots, \lambda a_n) &= G_n(\lambda\gamma|x_1 - y_1|, \ldots, \lambda\gamma\, |x_n - y_n|) \\ &= G_n(\gamma|\lambda x_1 - \lambda y_1|, \ldots, \gamma\, |\lambda x_n - \lambda y_n|) \\ &= D_n(\lambda x, \lambda y) \\ &= \lambda D_n(x,y) \\ &= \lambda G_n(\gamma|x_1 - y_1|, \ldots, \gamma\, |x_n - y_n|) \\ &= \lambda G_n(a_1, \ldots, a_n) \end{aligned}$$

and we are done. □

As noted earlier a total mobility measure should not decrease upon the addition of an individual to the population, and it should be unchanged if this additional person has not experienced any income change in the period under study. A slight strengthening of this idea is that, in the context of groups of varying sizes, "if equals are added to equals, the results are equal." This leads us to posit the following weak independence condition on the function sequence $\{D_n\}_{n \geq 1}$:

Axiom PC: **(Population Consistency)** Let $\{D_n\}_{n\geq 1} \in \prod_{n=1}^{\infty} \mathscr{D}^n$. For all $x, y \in \mathbb{R}^{n-1}_+$, $z, w \in \mathbb{R}^{n-2}_+$ and $a, b \geq 0$,

$$D_{n-1}(x,y) = D_{n-2}(z,w) \quad \text{implies} \quad D_n((x,a),(y,b)) = D_{n-1}((z,a),(w,b)) \ .$$

To illustrate this axiom, consider two populations of $n-1$ and $n-2$ individuals, respectively. Let $x \to y$ be observed in the first population and $z \to w$ be observed in the second one. Suppose that the level of income mobility is somehow judged to be the same in the two situations. Axiom PC says that if an *identical* agent (with initial income $a \geq 0$ and final income $b \geq 0$) is added in to both situations, then the two should still be judged to have the same mobility (i.e., $(x,a) \to (y,b)$ should be declared to exhibit the same level of mobility with $(z,a) \to (w,b)$). It seems to us that such a postulate is quite an appealing consistency requirement for total mobility indices.

That Axiom PC is in fact a separability condition is apparent from the following observation which will be quite useful when proving the main result of this section.

Lemma 3: Let $\{D_n\}_{n \geq 1} \in \prod_{n=1}^{\infty} \mathscr{D}^n$ *satisfy Axioms LH, TI, D and PC. Then, for any* $x, y \in \mathbb{R}_+^n$, $n \geq 3$,

$$D_n(x,y) = \gamma G_2(G_{n-1}(|x_1 - y_1|, \ldots, |x_{n-1} - y_{n-1}|), |x_n - y_n|)$$

for some $\gamma > 0$ *and some* $\{G_n\}_{n \geq 2}$ *which is a sequence of symmetric, positive, strictly increasing, continuous and linearly homogeneous functions on* $\mathbb{R}_+^n$.

Proof: Given Lemmas 1 and 2, this claim is virtually identical to Lemma 7.4 of Fields and Ok (1996); we omit the proof. □

Let us define, for any $n \geq 1$, $\gamma > 0$ and $\alpha \in [1, \infty)$, the function $D_n^{\alpha,\gamma} : \mathbb{R}_+^n \times \mathbb{R}_+^n \to \mathbb{R}_+$ as

$$D_n^{\alpha,\gamma}(x,y) := \gamma \left(\sum_{k=1}^{n} |x_k - y_k|^{\alpha} \right)^{1/\alpha} \quad \text{for all} \quad x, y \in \mathbb{R}_+^n . \tag{1}$$

The following theorem is the main result of this section.

Theorem 4: $\{D_n\}_{n \geq 1} \in \prod_{n=1}^{\infty} \mathscr{D}^n$ *satisfies Axioms LH, TI, D and PC if, and only if,*

$$\{D_n\}_{n \geq 1} = \{D_n^{\alpha,\gamma}\}_{n \geq 1} \quad \text{for some } \alpha \in [1, \infty) \text{ and } \gamma > 0 .$$

By Theorem 4, we observe that the four axioms discussed above are sufficient to characterize the following class of income mobility measures:

$$\mathscr{M} := \bigcup_{\alpha \in [1,\infty)} \bigcup_{\gamma > 0} \{\{D_n^{\alpha,\gamma}\}_{n \geq 1}\} . \tag{2}$$

As long as one views Axioms LH, TI, D, PC and GS as compelling, (s)he needs to use a mobility index of the form $\{D_n^{\alpha,\gamma}\}_{n \geq 1}$. We note that, since the choice of $\gamma > 0$ would not affect the mobility comparisons, the only degree of freedom is, in fact, in terms of choosing a specific $\alpha \in [1, \infty)$ (on which more shortly).

Before giving a proof of Theorem 4, let us mention that a member of $\mathscr{M}$ which deserves perhaps special attention is $\{D_n^{1,1}\}_{n \geq 1}$. This measure is uniquely characterized by Fields and Ok (1996) and is shown to be additively decomposable into two components; mobility due to the *transfer of income* within a given structure and mobility due to *economic growth*. We emphasize that such an exact decomposition (which is likely to be useful in empirical applications) does not appear in the case of $\{D_n^{\alpha,1}\}_{n \geq 1}$ when $\alpha > 1$.

We conclude this section by providing a

Proof of Theorem 4: That $\{D_n^{\alpha,\gamma}\}_{n \geq 1}$ for any $\gamma > 0$ and $\alpha \geq 1$ satisfies the stated axioms can easily be verified. We, therefore, focus only on the necessity part of the assertion. Assume that $\{D_n\}_{n \geq 1} \in \prod_{n=1}^{\infty} \mathscr{D}^n$ satisfies Axioms LH, TI, D and PC. Then by Lemma 3 and surjectivity of D_3 (guaranteed by Axiom LH), we have

$$G_3(a_1, a_2, a_3) = G_2(G_2(a_1, a_2), a_3) \quad \forall \, a_1, a_2, a_3 \geq 0 \tag{3}$$

where $G_k : \mathbb{R}_+^k \to \mathbb{R}_+$, $k = 2, 3$, are symmetric, strictly increasing, continuous and linearly homogeneous functions. By symmetry and Lemma 3, for all $a_1, a_2, a_3 \geq 0$,

$$G_3(a_1, a_2, a_3) = G_3(a_2, a_3, a_1) = G_2(G_2(a_2, a_3), a_1) = G_2(a_1, G_2(a_2, a_3))$$

and combining this with (3),

$$G_2(G_2(a_1, a_2), a_3) = G_2(a_1, G_2(a_2, a_3)) \quad \forall a_1, a_2, a_3 \geq 0 \ . \tag{4}$$

This teaches us that G_2 satisfies the *associativity equation* (Aczel (1966), pp. 253–72). On the other hand, by strict monotonicity of G_2, $G_2(., a)$ and $G_2(a, .)$ are injective on $\mathbb{R}_+$ for any $a \geq 0$. We can thus apply the theorem in Aczel (1966), p. 256, to get

$$G_2(a_1, a_2) = f(f^{-1}(a_1) + f^{-1}(a_2)) \quad \forall a_1, a_2 \geq 0 \ ,$$

for some strictly increasing and continuous function $f : \mathbb{R}_+ \to \mathbb{R}_+$; we conclude that G_2 is *quasi-linear* (Aczel, 1966, p. 151). Since G_2 is also linearly homogeneous, by Theorem 2.2.1 of Eichhorn (1978, p. 32), we must have either

$$G_2(a_1, a_2) = A a_1^r a_2^{1-r} \quad \forall a_1, \ a_2 > 0 \tag{5}$$

with $A > 0$ and $r \in (0, 1)$, or

$$G_2(a_1, a_2) = \left(\beta_1 a_1^\alpha + \beta_2 a_2^\alpha\right)^{1/\alpha} \quad \forall a_1, \ a_2 > 0 \tag{6}$$

with $\beta_1, \beta_2 > 0$ and $\alpha \in \mathbb{R}\backslash\{0\}$. But by Axiom D, $G_2(0, 0) = 0$ so that $f^{-1}(0) = f^{-1}(G_2(0, 0)) = 2f^{-1}(0)$, i.e., $f^{-1}(0) = f(0) = 0$. This, in turn, implies that $G_2(a_1, 0) = a_1$ for all $a_1 > 0$, and hence, (5) cannot hold. But (6) with $\alpha < 0$ cannot be true either, for otherwise we could not have G_2 continuous at the origin. Moreover, the symmetry of G_2 implies that $\beta_1 = \beta_2 = 1$. Finally, notice that if $\alpha \in (0, 1)$ and $G_2(a_1, a_2) = \left(a_1^\alpha + a_2^\alpha\right)^{1/\alpha}$ for all $a_1, a_2 \geq 0$, then Axiom D would imply that D_2 is not a distance function. Therefore, we must have

$$G_2(a_1, a_2) = \left(a_1^\alpha + a_2^\alpha\right)^{1/\alpha} \quad \forall a_1, \ a_2 > 0$$

where $\alpha \in [1, \infty)$.

The proof is completed by induction on n. Assume that

$$G_h(a_1, \ldots, a_h) = \left(\sum_{k=1}^{h} a_k^\alpha\right)^{1/\alpha} \quad \forall a_1, \ldots, a_h \geq 0 \ ,$$

where $\alpha \in [1, \infty)$ and $h \in \{2, 3, \ldots\}$. We have, by Lemma 3, the induction hypothesis, and the characterization of G_2,

$$\begin{aligned} G_{h+1}(a_1, \ldots, a_{h+1}) &= G_2(G_h(a_1, \ldots, a_h), a_{h+1}) \\ &= G_2\left(\left(\sum_{k=1}^{h} a_k^\alpha\right)^{1/\alpha}, a_{h+1}\right) = \left(\sum_{j=1}^{h+1} a_k^\alpha\right)^{1/\alpha} \end{aligned}$$

for any $a_1, \ldots, a_{k+1} \geq 0$. Hence, using Axiom D and Lemma 1, we may conclude that, for any $n \geq 1$,

$$D_n(x, y) = \gamma \left(\sum_{k=1}^{n} |x_k - y_k|^{\alpha} \right)^{1/\alpha} \quad \text{for all} \quad x, y \in \mathbb{R}_+^n$$

for some $\alpha \in [1, \infty)$ and $\gamma > 0$. The proof is complete. □

3 An income mobility ordering

In the previous section we have proposed $\mathcal{M}$ (see (2)) as a class of "reasonable" absolute income mobility measures. Consequently, given $x \to y$ and $z \to w$, where $x, y, z, w \in \mathbb{R}_+^n$, one who believes in Axioms LH, TI, D and PC must conclude that $x \to y$ is a "*more mobile*" process than $z \to w$ whenever

$$D_n^{\alpha,\gamma}(x, y) \geq D_n^{\alpha,\gamma}(z, w)$$

for a certain choice of $\{D_n^{\alpha,\gamma}\}_{n \geq 1} \in \mathcal{M}$, or equivalently, whenever

$$\sum_{k=1}^{n} |x_k - y_k|^{\alpha} \geq \sum_{k=1}^{n} |z_k - w_k|^{\alpha} \tag{7}$$

for a certain choice of $\alpha \in [1, \infty)$. But what could be the rationale for choosing one α value over another in practical applications? It is indeed quite difficult (if at all possible) to uncover the value judgement implied by a specific choice of $\alpha \geq 1$ to be used in (7). Since Theorem 4 is a full characterization, our axioms are certainly not of help with respect to this difficulty.

The problem is very similar to that of choosing a particular inequality index to evaluate the inequality of income distributions. The choice is quite consequential; it is well known that different inequality measures might result in drastically different rankings (see Champernowne, 1974; Kondor, 1975; Blackorby and Donaldson, 1978; Braun, 1988; *inter alia*.) Nevertheless, there is at least one way of making unambiguous inequality evaluations; if an income distribution Lorenz dominates another, then (and only then) we know that all (symmetric) relative inequality measures which satisfy Dalton's principle of equalizing transfers agree that the former distribution is less unequal than the latter.[7] Therefore, in making inequality comparisons, one should first check if the Lorenz dominance applies, and only if it does not, one should use a specific inequality measure. This indeed appears to be the actual practice.

We can, in fact, construct a similar device to make unambiguous income mobility comparisons in the present framework. Let us define the following binary relation on $\mathbb{R}_+^{2n}$: for all $x, y, z, w \in \mathbb{R}_+^n$, $n \geq 1$,

[7] This is undoubtedly a benchmark result in the theory of income inequality measurement. For extensive discussions, we refer the reader to Dasgupta et al. (1973), Sen (1973), Fields and Fei (1978), Foster (1985) and Jenkins (1991) among others.

$$(x,y) \succcurlyeq_{\mathrm{M}} (z,w) \text{ if and only if } D_n^{\alpha,\gamma}(x,y) \geq D_n^{\alpha,\gamma}(z,w) \text{ for all } \{D_n^{\alpha,\gamma}\}_{n\geq 1} \in \mathscr{M},$$

or equivalently,

$$(x,y) \succcurlyeq_{\mathrm{M}} (z,w) \text{ if and only if} \sum_{k=1}^{n} |x_k - y_k|^{\alpha} \geq \sum_{k=1}^{n} |z_k - w_k|^{\alpha} \text{ for all } \alpha \in [1,\infty) \ .$$

We define $\succ_{\mathrm{M}}$ as the asymmetric factor of $\succcurlyeq_{\mathrm{M}}$ as usual.[8]

The strengths and weaknesses of $\succcurlyeq_{\mathrm{M}}$ are very similar to the celebrated Lorenz ordering. Whenever it lets us order $x \to y$ and $z \to w$, the conclusion is agreed by all income mobility measures (defined as distance functions) that satisfy Axioms LH, TI, D, and PC. Therefore, when $\succcurlyeq_{\mathrm{M}}$ applies, the choice of α in (7) is immaterial; for all $\alpha \in [1,\infty)$, we shall obtain the same mobility ranking. Just like the Lorenz ordering, on the other hand, the drawback of $\succcurlyeq_{\mathrm{M}}$ clearly lies in its incompleteness.

Some basic properties of our absolute mobility ordering are readily observed: $\succcurlyeq_{\mathrm{M}}$ is reflexive, transitive and incomplete, it is therefore an incomplete preorder. Given $n \geq 1$, the set of least elements of $\mathbb{R}_+^{2n}$ with respect to $\succcurlyeq_{\mathrm{M}}$ is $\{(x,x) : x \in \mathbb{R}_+^n\}$. That is, $x \to y$ exhibits the least mobility with respect to $\succcurlyeq_{\mathrm{M}}$ if, and only if, each individual's income remains unchanged during the process; a highly intuitive conclusion. On the other extreme, one can easily see that the set of greatest elements of $\mathbb{R}_+^{2n}$ with respect to $\succcurlyeq_{\mathrm{M}}$ is the empty set. Since our notion of income mobility is sensitive to income growth, we view this implication too as reasonable.

In passing, we stress that defining our mobility ordering only for populations of the same size is, in fact, without loss of generality. Indeed, we could equivalently work with a mobility ordering induced by the per capita version of the class characterized in Theorem 4. To see this more clearly, let us take the class

$$\mathscr{M}^* := \bigcup_{\alpha\in[1,\infty)} \bigcup_{\gamma>0} \left\{ \left\{ \frac{D_n^{\alpha,\gamma}}{n} \right\}_{n\geq 1} \right\}$$

and define the ordering $\succcurlyeq_{\mathrm{M}}^*$ on $\cup_{k\geq 1}\mathbb{R}_+^{2k}$ as

$$(x,y) \succcurlyeq_{\mathrm{M}} (z,w) \text{ if and only if } M_n^{\alpha,\gamma}(x,y) \geq M_m^{\alpha,\gamma}(z,w) \text{ for all } \{M_n^{\alpha,\gamma}\}_{n\geq 1} \in \mathscr{M}^*$$

for all $x,y \in \mathbb{R}_+^n$ and $z,w \in \mathbb{R}_+^m$, $n,m \geq 1$. Clearly, $\succcurlyeq_{\mathrm{M}}^*$ is the per capita version of $\succcurlyeq_{\mathrm{M}}$ and is also axiomatically induced by Theorem 4. It is then this ordering that one would use to compare the mobility levels of two distributional transformations of different population sizes. Yet $\succcurlyeq_{\mathrm{M}}^*$ is actually fully characterized by $\succcurlyeq_{\mathrm{M}}$. Indeed, where $[a]_r$ denotes the r-fold replication of an object a, we have

[8] Our definition, of course, specifies rather a sequence of binary relations $\{\succcurlyeq_{\mathrm{M}}^n\}_{n\geq 1}$ where $\succcurlyeq_{\mathrm{M}}^n \subset \mathbf{R}_+^{2n} \times \mathbf{R}_+^{2n}$, $n \geq 1$. For brevity, we denote here $\succcurlyeq_{\mathrm{M}}^n$ by $\succcurlyeq_{\mathrm{M}}$ for any $n \geq 1$.

$$
\begin{aligned}
(x,y) \succcurlyeq^*_{\mathbf{M}} (z,w) &\Leftrightarrow M^{\alpha,\gamma}_{nm}([x]_m,[y]_m) \geq M^{\alpha,\gamma}_{nm}([z]_n,[w]_n) \quad \text{for all } \{M^{\alpha,\gamma}_n\}_{n\geq 1} \in \mathscr{M}^* \\
&\Leftrightarrow D^{\alpha,\gamma}_{nm}([x]_m,[y]_m) \geq D^{\alpha,\gamma}_{nm}([z]_n,[w]_n) \quad \text{for all } \{D^{\alpha,\gamma}_n\}_{n\geq 1} \in \mathscr{M} \\
&\Leftrightarrow ([x]_m,[y]_m) \succcurlyeq_{\mathbf{M}} ([z]_n,[w]_n)
\end{aligned}
$$

for all $x, y \in \mathbb{R}^n_+$ and $z, w \in \mathbb{R}^m_+$, $n, m \geq 1$. It must now be clear that convenience is the only reason why we confine our attention to comparing the income mobilities of populations of the same size.

4 Superrelations of $\succcurlyeq_{\mathbf{M}}$

Given the development of Sections 2 and 3, $\succcurlyeq_{\mathbf{M}}$ emerges as an interesting mobility ordering. The problem, of course, is that there is no way to check at the moment whether it orders a given (x,y) and (z,w) except in some trivial cases (like $z = w$). In this section, we shall develop some criteria to aid us determine when $\succcurlyeq_{\mathbf{M}}$ is actually *not* applicable. The converse question is taken up in the next section.

Let us start by introducing some notation. For any $x, y \in \mathbb{R}^n_+$, $n \geq 1$, define

$$\triangle(x,y) := \left(\left|x_{\sigma(1)} - y_{\sigma(1)}\right|, \ldots, \left|x_{\sigma(n)} - y_{\sigma(n)}\right|\right)$$

where $\sigma(.)$ is a permutation on $\{1,\ldots,n\}$ such that

$$\left|x_{\sigma(1)} - y_{\sigma(1)}\right| \geq \cdots \geq \left|x_{\sigma(n)} - y_{\sigma(n)}\right| \ ,$$

and let

$$\triangle_k(x,y) := \left|x_{\sigma(k)} - y_{\sigma(k)}\right| \quad \text{for all} \quad k \in \{1,\ldots,n\} \ .$$

In words, given $x \to y$, $\triangle(x,y)$ represents the vector of personal income changes which are ordered from largest to smallest, and $\triangle_k(x,y)$ denotes the kth largest amount of individual income change. We also define, for all $x, y, z, w \in \mathbb{R}^n_+$, $n \geq 1$,

$$\mathbf{A}_h(z,w,x,y) := \sum_{k=1}^{h} (\triangle_k(z,w) - \triangle_k(x,y)) \quad \text{for all} \quad h \in \{1,\ldots,n\} \ . \tag{8}$$

The following proposition gives some trivial necessary conditions for $\succcurlyeq_{\mathbf{M}}$ to be applicable, thereby teaching us something about the extent of incompleteness of this mobility ordering.

Proposition 5: *Let* $x, y, z, w \in \mathbb{R}^n_+$, $n \geq 1$. *If* $(x,y) \succcurlyeq_{\mathbf{M}} (z,w)$, *then we must have*

$$\mathbf{A}_1(z,w,x,y) \leq 0 \quad \textit{and} \quad \mathbf{A}_n(z,w,x,y) \leq 0 \ . \tag{9}$$

Proof: Fix $n \geq 1$ and let $(x,y) \succcurlyeq_{\mathbf{M}} (z,w)$, $x, y, z, w \in \mathbb{R}^n_+$. Then, by definition,

$$\sum_{k=1}^{n} ((\triangle_k(z,w))^\alpha - (\triangle_k(x,y))^\alpha) \leq 0 \quad \forall \alpha \in [1,\infty) \ . \tag{10}$$

The first statement in (9) follows from letting $\alpha \to \infty$ in this expression. The second statement is immediate upon setting $\alpha = 1$. $\square$

Example: Let $n = 3$, $x = (3, 5, 8)$, $y = (8, 15, 6)$, $z = (2, 10, 4)$ and $w = (8, 4, 10)$. Suppose that $x \to y$ and $z \to w$. Can we compare (x, y) and (z, w) by $\succcurlyeq_{\mathrm{M}}$? Notice that we have $\triangle(x, y) = (10, 5, 2)$ and $\triangle(z, w) = (6, 6, 6)$ so that $\mathbf{A}_1(z, w, x, y) = -4$ and $\mathbf{A}_3(x, y, z, w) = 1$. Therefore, in view of Proposition 5, (x, y) and (z, w) *cannot* be ranked by $\succcurlyeq_{\mathrm{M}}$.

A third necessary condition for $\succcurlyeq_{\mathrm{M}}$ to apply (that is, for (10) to hold) is somewhat less obvious, but provides a key insight which we shall exploit in the next section to develop appropriate sufficient conditions. To formulate this condition we need the following notation: For any $x, y \in \mathbb{R}^n_+, n \geq 1$,

$$\triangle_{k,k+1}(x, y) := \triangle_k(x, y) - \triangle_{k+1}(x, y) \quad \text{for all} \quad k \in \{1, \ldots, n-1\} \ .$$

Notice that, by definition of $\triangle(x, y)$, $\triangle_{k,k+1}(x, y) \geq 0$ for all $k \in \{1, \ldots, n-1\}$.

Proposition 6: *Let* $x, y, z, w \in \mathbb{R}^n_+$, $n \geq 1$. *If* $(x, y) \succcurlyeq_{\mathrm{M}} (z, w)$, *then we must have*

$$\sum_{k=1}^{n-1} \mathbf{A}_k(z, w, x, y) \triangle_{k,k+1}(x, y) + \mathbf{A}_n(z, w, x, y) \triangle_n(x, y) \leq 0 \ . \tag{11}$$

Proof: Fix $n \geq 1$ and $x, y, z, w \in \mathbb{R}^n_+$ such that $(x, y) \succcurlyeq_{\mathrm{M}} (z, w)$. By definition, (10) holds. Let

$$a_k := \triangle_k(x, y), \ \ b_k := \triangle_k(z, w) \quad \text{and} \quad \mathbf{A}_h := \mathbf{A}_h(z, w, x, y) = \sum_{k=1}^{h} (b_k - a_k)$$

for all $k, h \in \{1, \ldots, n\}$. By hypothesis, we have $\sum_{k=1}^n \left(b_k^\alpha - a_k^\alpha\right) \leq 0$ for all $\alpha \geq 1$; in particular, $\sum_{k=1}^n \left(b_k^2 - a_k^2\right) \leq 0$. By convexity of the mapping $t \mapsto t^2$, we have $b_k^2 - a_k^2 \geq 2a_k(b_k - a_k)$ for all $k \in \{1, \ldots, n\}$. Therefore,

$$\sum_{k=1}^n a_k(b_k - a_k) \leq \left(\frac{1}{2}\right) \sum_{k=1}^n \left(b_k^2 - a_k^2\right) \leq 0 \ .$$

But, by Abel's partial summation formula,

$$\sum_{k=1}^n a_k(b_k - a_k) = \mathbf{A}_1(a_1 - a_2) + \cdots + \mathbf{A}_{n-1}(a_{n-1} - a_n) + \mathbf{A}_n a_n$$

and (11) follows immediately. □

Example: Let $n = 3$, $x = (3, 5, 8)$, $y = (13, 8, 1)$, $z = (2, 10, 4)$ and $w = (11, 19, 6)$. Suppose that $x \to y$ and $z \to w$. Can we compare (x, y) and (z, w) by $\succcurlyeq_{\mathrm{M}}$? Here we have $\triangle(x, y) = (10, 7, 3)$ and $\triangle(z, w) = (9, 9, 2)$ so that (9) holds (and thus Proposition 5 is of no help). However, $\mathbf{A}_1(z, w, x, y) \triangle_{1,2}(x, y) + \mathbf{A}_2(z, w, x, y) \triangle_{2,3}(x, y) + \mathbf{A}_3(z, w, x, y) \triangle_3(x, y) = (-1)3 + 4 = 1$. Therefore, by Proposition 6, (x, y) and (z, w) *cannot* be ranked by $\succcurlyeq_{\mathrm{M}}$.

5 Subrelations of $\succcurlyeq_{\mathrm{M}}$

In this section we shall develop several sets of sufficient conditions for (10) to hold for all $x, y, z, w \in \mathbb{R}^n_+$. Since none of these conditions make a reference to

α, they may prove helpful in obtaining definitive conclusions when making absolute mobility comparisons by using $\succcurlyeq_{\mathrm{M}}$.

Our first theorem is readily deduced from the theory of majorization.

Theorem 7: *For any* $x, y, z, w \in \mathbb{R}^n_+$, $n \geq 1$, *if*

$$\sum_{k=1}^{h} \triangle_k(x,y) \geq \sum_{k=1}^{h} \triangle_k(z,w) \quad \textit{for all} \quad h \in \{1,\ldots,n\}\ , \tag{12}$$

then $(x,y) \succcurlyeq_{\mathrm{M}} (z,w)$.

Proof: The hypothesis of the theorem is equivalent to saying that $\triangle(x,y)$ *weakly submajorizes* $\triangle(z,w)$. Therefore, by the Tomić-Weil submajorization theorem (see, for instance, Marshall and Olkin, 1979, Proposition 4.B.2, p. 109), we have $\sum_{k=1}^{n} g(\triangle_k(x,y)) \geq \sum_{k=1}^{n} g(\triangle_k(z,w))$ for all continuous, increasing and convex functions $g : \mathbb{R}^n_+ \to \mathbb{R}$. But $t \mapsto t^\alpha$ defines a continuous, increasing and convex function on $\mathbb{R}_+$ for all $\alpha \geq 1$. Therefore, for all $\alpha \geq 1$,

$$\sum_{k=1}^{n} |x_k - y_k|^\alpha = \sum_{k=1}^{n} (\triangle_k(x,y))^\alpha \geq \sum_{k=1}^{n} (\triangle_k(z,w))^\alpha = \sum_{k=1}^{n} |z_k - w_k|^\alpha$$

and the theorem follows. □

Example: *Let* $n = 3$, $x = (3,5,9)$, $y = (13,8,1)$, $z = (2,10,4)$ and $w = (11,19,6)$. Suppose that $x \to y$ and $z \to w$. Can we compare (x,y) and (z,w) by $\succcurlyeq_{\mathrm{M}}$? Here we have $\triangle(x,y) = (10,8,3)$ and $\triangle(z,w) = (9,9,2)$. Since $\triangle(x,y)$ clearly submajorizes $\triangle(z,w)$ (recall (12)), by Theorem 7, we conclude that $(x,y) \succcurlyeq_{\mathrm{M}} (z,w)$.

As this example illustrates, submajorization relation provides a very easy way of applying our mobility ordering $\succcurlyeq_{\mathrm{M}}$. In fact, from Theorem 7 we learn that the incompleteness of $\succcurlyeq_{\mathrm{M}}$ is not that severe; it follows from this result that $\succcurlyeq_{\mathrm{M}}$ is not "more incomplete" than the submajorization ordering. But the submajorization relation is the dual of the supermajorization relation which is better known as the *second order stochastic dominance* or as the *generalized Lorenz ordering* in the economics literature. Thus, Theorem 7 teaches us that $\succcurlyeq_{\mathrm{M}}$ is not "more incomplete" than the generalized Lorenz ordering which is found to be very useful both in theory and practice (cf. Shorrocks, 1983).

In what follows, we shall state three theorems (all proved in the Appendix) which are ascending in strength. Put differently, in our next theorem, we will formulate a set of sufficient conditions (which are not implied by those of Theorem 7) for (10) to apply, and then in the consecutive result we shall show the sufficiency of a weaker set of conditions and so on. We hope that such a presentation will help clarify the intuition behind our final and the weakest set of sufficiency conditions, and may further illustrate how large (and useful) a subrelation of $\succcurlyeq_{\mathrm{M}}$ (which does not depend on α) we are presently able to uncover.

Before proceeding to formulate our sufficiency theorems, we need to introduce the following notation: For any $x, y, z, w \in \mathbb{R}^n_+$, $n \geq 1$, we write

$$\mathscr{P}(z, w, x, y) := \{k \in \{1, \ldots, n\} : \mathbf{A}_k(z, w, x, y) > 0\}$$

and define, for all $k \in \{1, \ldots, n-1\}$,

$$\mathbf{B}_k(z, w, x, y) := \begin{cases} \mathbf{A}_k(z, w, x, y) \frac{\triangle_1(z,w)}{\triangle_{k+1}(z,w)}, & \text{if } \mathbf{A}_k(z, w, x, y) > 0 \\ \mathbf{A}_k(z, w, x, y), & \text{if } \mathbf{A}_k(z, w, x, y) \leq 0 . \end{cases} \tag{13}$$

Here is our first result which complements Theorem 7.

Theorem 8: *For any* $x, y, z, w \in \mathbb{R}^n_+$, $n \geq 1$, *such that* $\triangle_n(z, w) > 0$, *if, for some* $\ell \in \{1, \ldots, n\}$,

$$\mathbf{A}_k(z, w, x, y) \leq 0 \quad \textit{for all} \quad k \in \{1, \ldots, \ell\} , \tag{14}$$

$$\mathbf{A}_n(z, w, x, y) \leq 0 , \tag{15}$$

and

$$\sum_{k=1}^{\ell} \mathbf{B}_k(z, w, x, y) \triangle_{k,k+1}(z, w) + \sum_{k \in \mathscr{P}(z,w,x,y)} \mathbf{B}_k(z, w, x, y) \triangle_{k,k+1}(z, w) \leq 0 , \tag{16}$$

then $(x, y) \succcurlyeq_{\mathrm{M}} (z, w)$.

Notice that if (12) holds and $\triangle_n(z, w) > 0$, then $\mathscr{P}(z, w, x, y) = \emptyset$, and hence all the conditions of Theorem 8 are clearly satisfied. Thus, the only reason why this result is not a strict refinement of Theorem 7 is due to the hypothesis $\triangle_n(z, w) > 0$.

The following example shows that Theorem 8 tells us something that Theorem 7 did not.

Example: Let $n = 3$, $x = (3, 5, 9)$, $y = (17, 8.5, 12)$, $z = (2, 10, 4)$ and $w = (14, 12.25, 10)$. Suppose that $x \rightarrow y$ and $z \rightarrow w$. Here we have $\triangle(x, y) = (14, 3.5, 3)$ and $\triangle(z, w) = (12, 6, 2.25)$ so that, letting $\mathbf{A}_k = \mathbf{A}_k(z, w, x, y)$, $k \in \{1, 2, 3\}$, $\mathbf{A}_1 = -2$, $\mathbf{A}_2 = 0.5$ and $\mathbf{A}_3 = -0.25$; Theorem 7 does not apply. Yet $\mathbf{B}_1 = \mathbf{A}_1$ and $\mathbf{B}_2 = 0.5(12/2.25) = 2.\bar{6}$ so that

$$\mathbf{B}_1 \triangle_{1,2}(z, w) + \mathbf{B}_2 \triangle_{2,3}(z, w) = (-2)6 + (2.\bar{6})(3.75) = -2.25 ,$$

that is, (16) holds and, by virtue of Theorem 8, we conclude that $(x, y) \succcurlyeq_{\mathrm{M}} (z, w)$.

To pave our way towards a stronger result, let us now look at condition (16) a bit more closely. Fix any $x, y, z, w \in \mathbb{R}^n_+$ such that $\triangle_n(z, w) > 0$, and (14) and (15) hold. Our analysis is based on the entries of the following vector

$$v = (\mathbf{B}_1 \triangle_{1,2}(z, w), \mathbf{B}_2 \triangle_{2,3}(z, w), \ldots, \mathbf{B}_{n-1} \triangle_{n-1,n}(z, w))$$

(where, of course, $\mathbf{B}_k = \mathbf{B}_k(z, w, x, y)$, $k \in \{1, \ldots, n-1\}$). Let us assume that $B_{\ell+1} > 0$ and recall that, by hypothesis, the first ℓ entries of this vector are negative. Condition (16) makes use of precisely these negative elements to outweigh the sum of all the positive entries of v. We might then be able to improve upon (16) by allowing *all* the negative elements of v to play a significant role in the sufficient condition. One way of doing this will be demonstrated in our next result.

Another shortcoming of Theorem 8 is that the precedent $\triangle_n(z,w) > 0$ is too stringent a hypothesis, for it is plausible that an individual's income level may stay the same in the transformation $z \to w$. (Recall that $\triangle_n(z,w)$ is the amount of the income change of the person who experiences the minimum degree of income change.) In our following result, this hypothesis too will be relaxed.

Before stating the next theorem, we define, for all $z, w \in \mathbb{R}^n_+$ such $z \neq w$,

$$q(z,w) := \begin{cases} \max\{k \in \{1,\ldots,n\} : \triangle_k(z,w) > 0\}, & \text{if } \triangle_n(z,w) = 0 \\ n, & \text{if } \triangle_n(z,w) > 0 \end{cases} .$$

That is, $\triangle_{q(z,w)}(z,w)$ is the *smallest* non-zero individual income change that is observed in the process $z \to w$.

The following strengthening of Theorem 8 is true:

Theorem 9: *Let $n \geq 1$ and take any $x, y, z, w \in \mathbb{R}^n_+$ such that $z \neq w$. If*

$$\mathbf{A}_{q(z,w)}(z,w,x,y) \leq 0 \ , \tag{17}$$

and

$$\sum_{k=1}^{s} \mathbf{B}_k(z,w,x,y)\triangle_{k,k+1}(z,w) \leq 0 \quad \textit{for all} \quad s \in \{1,\ldots,q(z,w)-1\} \ , \tag{18}$$

then $(x,y) \succcurlyeq_{\mathbf{M}} (z,w)$.

That Theorem 9 is indeed a generalization of Theorem 8 is shown next.

Example: *Let* $n = 5$, $x = (3,5,9,10,8)$, $y = (5.5,7.5,2,17,36)$, $z = (2,10,4,8,13)$ and $w = (13,35,0,10,18)$. Suppose that $x \to y$ and $z \to w$. Here we have $\triangle(x,y) = (28,7,7,2.5,2.5)$ and $\triangle(z,w) = (25,11,5,4,2)$ so that $\mathbf{A}_1 = -3$, $\mathbf{A}_2 = 1$, $\mathbf{A}_3 = -1$, $\mathbf{A}_4 = 0.5$ and $\mathbf{A}_5 = 0$. That Theorem 7 does not apply is obvious. Also, $\mathbf{B}_1 = -3$, $\mathbf{B}_2 = 5$, $\mathbf{B}_3 = -1$ and $\mathbf{B}_4 = 0.5(25/2) = 6.25$ so that

$$\begin{aligned} &\mathbf{B}_1\triangle_{1,2}(z,w) + \mathbf{B}_2\triangle_{2,3}(z,w) + \mathbf{B}_4\triangle_{4,5}(z,w) \\ &\quad = (-3)14 + 5(6) + (6.25)(2) = 0.5 \ . \end{aligned}$$

Thus, (16) does not hold, and we cannot make use of Theorem 8 either. However, we find here that $q(z,w) = 5$ and that

$$\begin{aligned} &\mathbf{B}_1\triangle_{1,2}(z,w) = -42 \\ &\mathbf{B}_1\triangle_{1,2}(z,w) + \mathbf{B}_2\triangle_{2,3}(z,w) = -12 \\ &\mathbf{B}_1\triangle_{1,2}(z,w) + \mathbf{B}_2\triangle_{2,3}(z,w) + \mathbf{B}_3\triangle_{3,4}(z,w) = -13 \\ &\mathbf{B}_1\triangle_{1,2}(z,w) + \mathbf{B}_2\triangle_{2,3}(z,w) + \mathbf{B}_3\triangle_{3,4}(z,w) + \mathbf{B}_4\triangle_{4,5}(z,w) = -0.5 \ , \end{aligned}$$

that is, (17) and (18) hold. Therefore, by Theorem 9, we conclude that $(x,y) \succcurlyeq_{\mathbf{M}} (z,w)$.

In our final result, we shall formulate a refinement of Theorem 8 in the hope of increasing the applicability of our income mobility ordering $\succcurlyeq_{\mathbf{M}}$. The basic idea behind the refinement comes from the observation that while (18) uses all the negative entries of the vector v to outweigh $\sum_{k\in\mathscr{P}(z,w,x,y)} \mathbf{B}_k \triangle_{k,k+1}(z,w)$, we are, by virtue of (17), endowed with a further negative term: $\mathbf{B}_{q(z,w)} \leq 0$. Given (17), one might then be able to exploit the magnitude of $\mathbf{B}_{q(z,w)}$ to ensure that a smaller positive number replaces $\sum_{k\in\mathscr{P}(z,w,x,y)} \mathbf{B}_k \triangle_{k,k+1}(z,w)$ in the sufficiency condition (18). This idea is formalized next.

Once again we need to introduce some notation before stating the theorem. Let us define, for any $x, y, z, w \in \mathbb{R}^n_+$ such that $z \neq w$,

$$\eta := \begin{cases} 1 - \dfrac{|\mathbf{A}_q|\Delta_n(z,w)}{\sum\limits_{\substack{k\in\mathscr{P}\\ k\leq q-1}} \mathbf{A}_k \Delta_{k,k+1}(z,w)}, & \text{if } \sum\limits_{\substack{k\in\mathscr{P}\\ k\leq q-1}} \mathbf{A}_k \Delta_{k,k+1}(z,w) + \mathbf{A}_q \Delta_q(z,w) > 0 \\ 0, & \text{otherwise} \end{cases}$$

where $\mathbf{A}_k := \mathbf{A}_k(z,w,x,y)$ for all $k \in \{1,\ldots,n-1\}$, $q = q(z,w)$ and $\mathscr{P} = \mathscr{P}(z,w,x,y)$. (Of course, η is a function of x, y, z and w, but for expositional clarity we do not use a notation which makes this explicit.) We also define, for any $x, y, z, w \in \mathbb{R}^n_+$,

$$\mathbf{C}_k(z,w,x,y) := \begin{cases} \mathbf{A}_k(z,w,x,y)\left(\frac{\Delta_1(z,w)}{\Delta_{k+1}(z,w)}\right)^{\eta}, & \text{if } \mathbf{A}_k(z,w,x,y) > 0 \\ \mathbf{A}_k(z,w,x,y), & \text{if } \mathbf{A}_k(z,w,x,y) \leq 0\,. \end{cases}$$

We can now state

Theorem 10: *Let $x, y, z, w \in \mathbb{R}^n_+$, $n \geq 1$, and $z \neq w$. If*

$$\mathbf{A}_{q(z,w)}(z,w,x,y) \leq 0 \ ,$$

and

$$\sum_{k=1}^{s} \mathbf{C}_k(z,w,x,y) \triangle_{k,k+1}(z,w) \leq 0 \quad \textit{for all} \quad s \in \{1,\ldots,q(z,w)-1\} \ , \tag{19}$$

then $(x,y) \succcurlyeq_{\mathbf{M}} (z,w)$.

Notice that by definition, $0 \leq \eta \leq 1$ for any $x, y, z, w \in \mathbb{R}^n_+$. Therefore, for all $k \in \{1,\ldots n-1\}$, we have

$$\mathbf{B}_k(z,w,x,y) \geq \mathbf{C}_k(z,w,x,y) \geq \mathbf{A}_k(z,w,x,y) \ .$$

It is in this sense Theorem 10 is a generalization of Theorem 9. Here is a concrete example which highlights the contribution of Theorem 10 over Theorems 7, 8 and 9.

Example: Let $n = 3$, $x = (3,5,9)$, $y = (54,13,16)$, $z = (2,10,4)$ and $w = (52,20,2)$. Suppose that $x \to y$ and $z \to w$. Here we have $\triangle(x,y) = (51,8,7)$ and $\triangle(z,w) = (50,10,2)$ so that Theorem 7 does not apply. Also, $q(z,w) = 3$, $\mathbf{A}_1 = -1$ and $\mathbf{A}_2 = 1$ so that $\mathbf{B}_1 = -1$ and $\mathbf{B}_2 = 25$ and

$$\mathbf{B}_1 \triangle_{1,2}(z,w) + \mathbf{B}_2 \triangle_{2,3}(z,w) = (-1)(40) + 25(8) = 160 \ .$$

One observes that neither Theorem 8 nor Theorem 9 can be used to make a mobility comparison between the processes $x \to y$ and $z \to w$. Yet $\mathbf{A}_3 = -4$ and

$$\mathbf{A}_2 \triangle_{2,3}(z,w) + \mathbf{A}_3 \triangle_3(z,w) = 8 - (4)2 = 0$$

so that $\eta = 0$. Therefore, $\mathbf{C}_1 = \mathbf{A}_1 = -1$ and $\mathbf{C}_2 = \mathbf{A}_2 = 1$, and

$$\mathbf{C}_1 \triangle_{1,2}(z,w) + \mathbf{C}_2 \triangle_{2,3}(z,w) = (-1)(40) + 8 = -32 \ .$$

But then (17) and (19) are satisfied, and by Theorem 10, we can conclude that $(x,y) \succcurlyeq_{\text{M}} (z,w)$.

The above results, therefore, provide us with different sets of conditions which are computationally easy to check and which guarantee the satisfaction of (10) for any $x,y,z,w \in \mathbb{R}^n_+$, $n \geq 1$. Since we have demonstrated in Sections 2 and 3 that there is ample reason to conclude that $x \to y$ is a more mobile process than $z \to w$ when (10) holds (i.e. when $(x,y) \succcurlyeq_{\text{M}} (z,w)$), we believe that these results may prove quite useful in empirical applications.

6 Conclusion

In this paper, we have explored the implications of four axioms for absolute income mobility measures. We then showed that a structurally familiar class of measures is characterized by these axioms. Consequently, defining the following partial mobility ordering (of transformations of income distributions) is natural: a given transformation is "*more mobile*" than another if, and only if, the former is ranked higher than the latter for *all* mobility measures belonging to the characterized class.

The intuitive support of the proposed mobility ordering, $\succcurlyeq_{\text{M}}$, is similar to that of the Lorenz ordering. Whenever a transformation is ranked higher by this ordering than another, there is a clear sense in which one may conclude that the former process is *unambiguously* more mobile than the latter.

Due to its continuous dependence on a parameter, however, it was not readily apparent how one can apply our mobility ordering to panel data directly. The second part of our research was, therefore, directed towards overcoming this difficulty. As a result, we have obtained several sets of necessary and sufficient conditions which are very easy to check and which let one apply $\succcurlyeq_{\text{M}}$ to certain longitudinal data sets. Regarding the empirical applications, one would, of course, like to obtain a complete characterization of the proposed mobility ordering without making any reference to a parameter value. The present study admittedly falls short of reaching such a characterization result. Naturally, this will be the subject of future research.

7 Appendix: Proofs of Theorems 8, 9, and 10

Throughout this appendix, we shall simplify our notation by writing

$$a_k := \Delta_k(x,y), \quad b_k := \Delta_k(z,w), \quad \mathscr{P} := \mathscr{P}(z,w,x,y), \quad q := q(z,w) \ ,$$

$$\mathbf{A}_h := \mathbf{A}_h(z,w,x,y) = \sum_{k=1}^{h}(b_k - a_k), \ \ \mathbf{B}_k := \mathbf{B}_k(z,w,x,y) = \begin{cases} \mathbf{A}_k \frac{b_1}{b_{k+1}}, & \text{if } \mathbf{A}_k > 0 \\ \mathbf{A}_k, & \text{if } \mathbf{A}_k \leq 0 \ . \end{cases}$$

and

$$\mathbf{C}_k := \mathbf{C}_k(z,w,x,y) = \begin{cases} \mathbf{A}_k \left(\frac{b_1}{b_{k+1}}\right)^{\eta}, & \text{if } \mathbf{A}_k > 0 \\ \mathbf{A}_k, & \text{if } \mathbf{A}_k \leq 0 \end{cases}$$

for all $k, h \in \{1, \ldots, n\}$, for any given $x, y, z, w \in \mathbb{R}^n_+$, $n \geq 1$. The following lemma will be used in all three of the subsequent proofs.

Lemma 11: *For any* $x, y, z, w \in \mathbb{R}^n_+$, $n \geq 1$, *if*

$$\sum_{k=1}^{q-1} \mathbf{A}_k(b_k^{\alpha-1} - b_{k+1}^{\alpha-1}) + \mathbf{A}_q b_q^{\alpha-1} \leq 0 \quad \textit{for all} \quad \alpha \in [1,\infty) \tag{20}$$

then, $(x,y) \succcurlyeq_{\mathrm{M}} (z,w)$.

Proof: By Abel's partial summation formula, for all $\alpha \in [1,\infty)$,

$$\sum_{k=1}^{n} b_k^{\alpha-1}(b_k - a_k) = \sum_{k=1}^{q-1} \mathbf{A}_k(b_k^{\alpha-1} - b_{k+1}^{\alpha-1}) + \mathbf{A}_q b_q^{\alpha-1} \ . \tag{21}$$

But since $t \mapsto t^\alpha$ is a convex mapping on $\mathbb{R}_+$ for all $\alpha \geq 1$, we have $b_k^\alpha - a_k^\alpha \leq \alpha b_k^{\alpha-1}(b_k - a_k)$ for all $\alpha \in [1,\infty)$. Therefore, by summing over k and combining the outcome with (21), we have

$$\sum_{k=1}^{n}(b_k^\alpha - a_k^\alpha) \leq \alpha\left(\sum_{k=1}^{q-1} \mathbf{A}_k(b_k^{\alpha-1} - b_{k+1}^{\alpha-1}) + \mathbf{A}_q b_q^{\alpha-1}\right)$$

and this proves the lemma. □

We now proceed to prove Theorems 8, 9 and 10.

Proof of Theorem 8: Fix $n \geq 1, \ell \in \{1, \ldots, n\}$ and $x, y, z, w \in \mathbb{R}^n_+$ such that $q = n$, and (14), (15) and (16) hold. We wish to show that these hypotheses imply (20) (with $q = n$), for we will then be done by Lemma 11. We distinguish between two cases.

Case 1: $\alpha \in [1, 2)$.

In this case, $t \mapsto t^{\alpha-1}$ is a concave mapping on $\mathbb{R}_+$, and hence, for $k \in \{\ell+1, \ldots, n-1\}$,

$$b_k^{\alpha-1} - b_{k+1}^{\alpha-1} \leq (\alpha-1)b_{k+1}^{\alpha-2}(b_k - b_{k+1}) = (\alpha-1)\left(\frac{b_1}{b_{k+1}}\right)^{2-\alpha} b_1^{\alpha-2}(b_k - b_{k+1}) \ ,$$

and since $b_1 \geq b_{k+1}$ for all $k \in \{\ell, \ldots, n-1\}$ and $\alpha \in [1,2)$, we conclude that

$$b_k^{\alpha-1} - b_{k+1}^{\alpha-1} \leq (\alpha-1)\left(\frac{b_1}{b_{k+1}}\right) b_1^{\alpha-2}(b_k - b_{k+1}) \quad \forall k \in \{\ell+1, \ldots, n-1\} \ . \tag{22}$$

(Since $b_n > 0$ by hypothesis, these inequalities are well-defined.) Also by concavity of $t \mapsto t^{\alpha-1}$,

$$b_k^{\alpha-1} - b_{k+1}^{\alpha-1} \geq (\alpha-1)b_k^{\alpha-2}(b_k - b_{k+1}) \quad \forall k \in \{1, \ldots, \ell\} \ . \tag{23}$$

But then, since $\mathbf{A}_k \leq 0$ for all $k \notin \mathscr{P}$, $\mathbf{A}_1, \ldots, \mathbf{A}_\ell$, $\mathbf{B}_1, \ldots, \mathbf{B}_\ell \leq 0$, and $(b_k/b_1)^{\alpha-2} \geq 1$ for each $k \in \{1, \ldots, \ell\}$, (23), (22) and (16) imply that

$$\begin{aligned}
&\left(\frac{1}{b_1}\right)^{\alpha-2}\left(\sum_{k=1}^{n-1} \mathbf{A}_k(b_k^{\alpha-1} - b_{k+1}^{\alpha-1})\right) \\
&\quad \leq (\alpha-1)\sum_{k=1}^{\ell} \mathbf{A}_k\left(\frac{b_k}{b_1}\right)^{\alpha-2}(b_k - b_{k+1}) + (\alpha-1)\sum_{k\in\mathscr{P}} \mathbf{A}_k\left(\frac{b_1}{b_{k+1}}\right)(b_k - b_{k+1}) \\
&\quad \leq (\alpha-1)\sum_{k=1}^{\ell} \mathbf{B}_k(b_k - b_{k+1}) + (\alpha-1)\sum_{k\in\mathscr{P}} \mathbf{B}_k(b_k - b_{k+1}) \leq 0 \ .
\end{aligned}$$

Since $\mathbf{A}_n \leq 0$ by (15), we conclude that (20) holds.

Case 2: $\alpha \in [2, \infty)$.

In this case, $t \mapsto t^{\alpha-1}$ defines a convex function on $\mathbb{R}_+$, and therefore, for any $k \in \{1, \ldots, n-1\}$,

$$b_k^{\alpha-1} - b_{k+1}^{\alpha-1} \leq (\alpha-1)b_k^{\alpha-2}(b_k - b_{k+1}) \ .$$

But, by definition, $b_{\ell+1} \geq b_k > 0$ for each $k \in \{\ell+1, \ldots, n-1\}$, and therefore,

$$b_k^{\alpha-1} - b_{k+1}^{\alpha-1} \leq (\alpha-1)b_{\ell+1}^{\alpha-2}(b_k - b_{k+1}) \quad \forall k \in \{\ell+1, \ldots, n-1\} \ . \tag{24}$$

Also, again by convexity of $t \mapsto t^{\alpha-1}$ on $\mathbb{R}_+$,

$$b_k^{\alpha-1} - b_{k+1}^{\alpha-1} \geq (\alpha-1)b_{k+1}^{\alpha-2}(b_k - b_{k+1}) \quad \forall k \in \{1, \ldots, \ell\} \ . \tag{25}$$

But then, since $\mathbf{B}_k \geq \mathbf{A}_k$ for all $k \in \{1, \ldots, n-1\}$, $\mathbf{A}_k \leq 0$ for all $k \notin \mathscr{P}$, $\mathbf{A}_1, \ldots, \mathbf{A}_\ell$, $\mathbf{B}_1, \ldots, \mathbf{B}_\ell \leq 0$, and $(b_{k+1}/b_{\ell+1})^{\alpha-2} \geq 1$ for each $k \in \{1, \ldots, \ell\}$, (25), (24) and (16) imply that

$$\begin{aligned}
&\left(\frac{1}{b_{\ell+1}}\right)^{\alpha-2}\left(\sum_{k=1}^{n-1} \mathbf{A}_k(b_k^{\alpha-1} - b_{k+1}^{\alpha-1})\right) \\
&\quad \leq (\alpha-1)\sum_{k=1}^{\ell} \mathbf{B}_k\left(\frac{b_{k+1}}{b_{\ell+1}}\right)^{\alpha-2}(b_k - b_{k+1}) + (\alpha-1)\sum_{k\in\mathscr{P}} \mathbf{B}_k(b_k - b_{k+1}) \\
&\quad \leq (\alpha-1)\sum_{k=1}^{\ell} \mathbf{B}_k(b_k - b_{k+1}) + (\alpha-1)\sum_{k\in\mathscr{P}} \mathbf{B}_k(b_k - b_{k+1}) \leq 0 \ .
\end{aligned}$$

Since $\mathbf{A}_n \leq 0$ by (15), (20) follows. □

Proof of Theorem 9: Fix $n \geq 1$, and take any $x, y, z, w \in \mathbb{R}^n_+$ such that $z \neq w$ (i.e. $q \geq 1$), and (17) and (18) hold. We shall show that these hypotheses imply (20) again by distinguishing between two cases.

Case 1: $\alpha \in [1, 2)$.

Define

$$\mathcal{N} := \{k \in \{1, \ldots, n-1\} : \mathbf{A}_k < 0\} \ ,$$

and notice that, by concavity of the mapping $t \mapsto t^{\alpha-1}$ on $\mathbb{R}_+$ and the fact that $b_1/b_{k+1} \geq 1$ for all k, we have

$$b_k^{\alpha-1} - b_{k+1}^{\alpha-1} \leq (\alpha - 1)\left(\frac{b_1}{b_{k+1}}\right) b_1^{\alpha-2}(b_k - b_{k+1}) \quad \forall k \in \mathcal{P} \cap \{1, \ldots, q-1\}$$

and

$$b_k^{\alpha-1} - b_{k+1}^{\alpha-1} \geq (\alpha - 1) b_1^{\alpha-2}(b_k - b_{k+1}) \quad \forall k \in \mathcal{N} \ .$$

The rest of the proof is analogous to the corresponding case of the proof of Theorem 8.

Case 2: $\alpha \in [2, \infty)$.

By using the convexity of $t \mapsto t^{\alpha-1}$ on $\mathbb{R}_+$, we have

$$b_k^{\alpha-1} - b_{k+1}^{\alpha-1} \leq (\alpha - 1) b_k^{\alpha-2}(b_k - b_{k+1}) \quad \forall k \in \mathcal{P}$$

and

$$b_k^{\alpha-1} - b_{k+1}^{\alpha-1} \geq (\alpha - 1) b_{k+1}^{\alpha-2}(b_k - b_{k+1}) \quad \forall k \in \mathcal{N}.$$

Therefore, defining

$$c_k := \begin{cases} b_k^{\alpha-2}, & \text{if } k \in \mathcal{P} \\ b_{k+1}^{\alpha-2}, & \text{if } k \in \mathcal{N} \end{cases} ,$$

we have

$$\begin{aligned} \sum_{k=1}^{n-1} \mathbf{A}_k (b_k^{\alpha-1} - b_{k+1}^{\alpha-1}) &\leq (\alpha - 1)\left(\sum_{k \in \mathcal{N}} \mathbf{A}_k b_{k+1}^{\alpha-2}(b_k - b_{k+1}) + \sum_{k \in \mathcal{P}} \mathbf{A}_k b_k^{\alpha-2}(b_k - b_{k+1})\right) \\ &= (\alpha - 1) \sum_{k=1}^{q-1} c_k \mathbf{A}_k (b_k - b_{k+1}) \\ &\leq \sum_{k=1}^{q-1} (\alpha - 1) c_k \mathbf{B}_k (b_k - b_{k+1}) \end{aligned} \tag{26}$$

(The last inequality follows from the fact that $\mathbf{B}_k \geq \mathbf{A}_k$ for all $k \in \{1, \ldots, q-1\}$.) Now let

$$\mathbf{T} := \max_{s \in \{1, \ldots, q-1\}} \sum_{k=1}^{s} (\alpha - 1) \mathbf{B}_k (b_k - b_{k+1}) \ .$$

Since one can easily verify that $c_1 \geq \cdots \geq c_{q-1} > 0$, we can apply Abel's inequality[9] and conclude that

$$\sum_{k=1}^{q-1} (\alpha - 1) c_k \mathbf{B}_k (b_k - b_{k+1}) \leq \mathbf{T} c_1 \ .$$

But by (18), $\mathbf{T} \leq 0$ and hence, in view of (26) and (17), (20) is established. □

Proof of Theorem 10: Fix $n \geq 1$, and take any $x, y, z, w \in \mathbb{R}^n_+$ such that $z \neq w$ (i.e. $b_1 > 0$), and (17) and (19) hold. Once again we wish to show that these hypotheses imply (20). If $\alpha \in [2, \infty)$,the analysis of case 2 of the proof of Theorem 9 goes through by replacing $\mathbf{B}_k$s by $\mathbf{C}_k$s. (Recall that $\mathbf{C}_k \geq \mathbf{A}_k$ for all $k \in \{1, \ldots, n-1\}$.) So let us assume that $\alpha \in [1, 2)$.

As in case 1 of the proof of Theorem 8,

$$b_k^{\alpha-1} - b_{k+1}^{\alpha-1} \leq (\alpha - 1) \left(\frac{b_1}{b_{k+1}} \right)^{2-\alpha} b_1^{\alpha-2} (b_k - b_{k+1}) \quad \forall k \in \mathscr{P} \cap \{1, \ldots, q-1\} \ ,$$

and

$$b_k^{\alpha-1} - b_{k+1}^{\alpha-1} \geq (\alpha - 1) b_k^{\alpha-2} (b_k - b_{k+1}) \quad \forall k \in \mathscr{N} \ .$$

Therefore,

$$\begin{aligned}
&\left(\frac{1}{b_1} \right)^{\alpha-2} \left(\sum_{k=1}^{n-1} \mathbf{A}_k (b_k^{\alpha-1} - b_{k+1}^{\alpha-1}) + \mathbf{A}_n b_n^{\alpha-1} \right) \\
&\quad = \left(\frac{1}{b_1} \right)^{\alpha-2} \left(\sum_{k=1}^{q-1} \mathbf{A}_k (b_k^{\alpha-1} - b_{k+1}^{\alpha-1}) + \mathbf{A}_q b_q^{\alpha-1} \right) \\
&\quad\quad \leq (\alpha - 1) \left(\sum_{\substack{k \in \mathscr{N} \\ k \leq q-1}} \mathbf{A}_k \left(\frac{b_k}{b_1} \right)^{\alpha-2} (b_k - b_{k+1}) + \sum_{\substack{k \in \mathscr{P} \\ k \leq q-1}} \mathbf{A}_k \left(\frac{b_1}{b_{k+1}} \right)^{2-\alpha} (b_k - b_{k+1}) \right) \\
&\quad\quad + \mathbf{A}_q b_q \left(\frac{b_1}{b_q} \right)^{2-\alpha} \\
&\quad\quad \leq (\alpha - 1) \left(\sum_{\substack{k \in \mathscr{N} \\ k \leq q-1}} \mathbf{A}_k (b_k - b_{k+1}) + \sum_{\substack{k \in \mathscr{P} \\ k \leq q-1}} \mathbf{A}_k \left(\frac{b_1}{b_{k+1}} \right)^{2-\alpha} (b_k - b_{k+1}) \right) \\
&\quad\quad + \mathbf{A}_q b_q \left(\frac{b_1}{b_q} \right)^{2-\alpha}
\end{aligned} \tag{27}$$

[9] *Abel's Inequality*: For any real numbers $u_1, \ldots, u_m$ and $v_1, \dot{s}, v_m$ such that $v_1 \geq \cdots \geq v_m \geq 0$,

$$\left(\min_{s \in \{1, \ldots, m\}} \sum_{k=1}^{s} u_k \right) v_1 \leq \sum_{k=1}^{m} u_k v_k \leq \left(\max_{s \in \{1, \ldots, m\}} \sum_{k=1}^{s} u_k \right) v_1 \ .$$

Proof: See Mitrinović (1970, p. 32, Theorem 2.2.1).

(The last step follows from the fact that $\mathbf{A}_k \leq 0$ and $\left(\frac{b_k}{b_1}\right)^{\alpha-2} \geq 1$ for $\alpha \in [1,2)$ and all $k \in \mathcal{N}$.) Notice that if

$$(\alpha - 1) \sum_{\substack{k \in \mathcal{P} \\ k \leq q-1}} \mathbf{A}_k (b_k - b_{k+1}) + \mathbf{A}_n b_n \leq 0 \ ,$$

in view of (27) and the fact that $(b_1/b_{k+1}) \leq (b_1/b_n)$ for all $k \in \{2, \ldots, q-1\}$, we observe that (20) is satisfied. So assume that

$$(\alpha - 1) \sum_{\substack{k \in \mathcal{P} \\ k \leq q-1}} \mathbf{A}_k (b_k - b_{k+1}) + \mathbf{A}_n b_n > 0 \tag{28}$$

Then, recalling that $\mathbf{A}_q \leq 0$ by (17),

$$\alpha - 1 > \frac{|\mathbf{A}_n| b_n}{\sum_{\substack{k \in \mathcal{P} \\ k \leq q-1}} \mathbf{A}_k (b_k - b_{k+1})} \ . \tag{29}$$

But by (28) and the hypothesis that $\alpha \in [1,2)$, we must have $\sum_{\substack{k \in \mathcal{P} \\ k \leq q-1}} \mathbf{A}_k (b_k - b_{k+1}) + \mathbf{A}_q b_q > 0$ which implies that

$$\eta = 1 - \frac{|\mathbf{A}_q| b_q}{\sum_{\substack{k \in \mathcal{P} \\ k \leq q-1}} \mathbf{A}_k (b_k - b_{k+1})} \ .$$

Therefore, by (29), $\alpha - 1 > 1 - \eta$. But then $2 - \alpha < \eta$ and this yields

$$\left(\frac{b_1}{b_{k+1}}\right)^{2-\alpha} \leq \left(\frac{b_1}{b_{k+1}}\right)^{\eta} \quad \forall k \in \mathcal{P} \cap \{1, \ldots, q-1\} \ .$$

Thus, using (27) and discarding the term involving $\mathbf{A}_q$, we get

$$\begin{aligned}
&\left(\frac{1}{b_1}\right)^{\alpha-2} \left(\sum_{k=1}^{n-1} \mathbf{A}_k (b_k^{\alpha-1} - b_{k+1}^{\alpha-1}) + \mathbf{A}_n b_n^{\alpha-1} \right) \\
&\qquad \leq (\alpha - 1) \sum_{\substack{k \in \mathcal{N} \\ k \leq q-1}} \mathbf{A}_k (b_k - b_{k+1}) + (\alpha - 1) \sum_{\substack{k \in \mathcal{P} \\ k \leq q-1}} \mathbf{A}_k \left(\frac{b_1}{b_{k+1}}\right)^{2-\alpha} (b_k - b_{k+1}) \\
&\qquad \leq (\alpha - 1) \sum_{\substack{k \in \mathcal{N} \\ k \leq q-1}} \mathbf{A}_k (b_k - b_{k+1}) + (\alpha - 1) \sum_{\substack{k \in \mathcal{P} \\ k \leq q-1}} \mathbf{A}_k \left(\frac{b_1}{b_{k+1}}\right)^{\eta} (b_k - b_{k+1}) \\
&\qquad = (\alpha - 1) \sum_{k=1}^{q-1} \mathbf{C}_k (b_k - b_{k+1})
\end{aligned}$$

Therefore, by (19), (20) is satisfied and, in view of Lemma 11, the proof is complete. □

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